Auction Theory

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This paper provides a brief introduction to auction theory. It derives equilibrium bidding for both the Dutch and the English auction (two auctions that are commonly used in practice) and computes for these two auctions the expected revenue for the seller. Both auctions appear to be revenue equivalent. It is shown that this is not a coincidence: any auction that allocates the object to the bidder with the highest value (provided that this value exceeds a certain threshold value) yields the same expected revenue. The Dutch and the English auction are revenue maximizing if the seller imposes the correct reserve price.

Introduction

Auctions have been widely used over thousands of years. The Babylonians auctioned wives, the ancient Greeks sold mine concessions in auctions, and the Romans put slaves, war booty, and debtors property up for auction, among many other things[1]. Nowadays, the use of auctions is also widespread. There are auctions for art, fish, flowers, and wine, but also for more abstract objects like treasury bills, radio frequency spectrum, and electricity distribution contracts. In some of these auctions, the amount of money raised is almost beyond imagination. In the 1990s, the US government collected tens of billions of dollars in auctions for licenses for second generation mobile telecommunication[2], and in 2000, both the British and German governments raised tens of billions of euros in auctions for license or third generation mobile telecommunication[3]. Auction theorists were closely involved in several of these auctions, both consulting governments on the designs of these auctions and advising bidders on their bidding strategies.This has generated a burst of auction theory.

Auction theory is a collection of game-theoretic models related to the interaction of bidders in auctions, and was pioneered by William Vickrey in 1961[4]. Vickrey, an economist from the Columbia University in New York, studied private value auctions, in which each bidder's value for the object for sale is independent of the values of the other bidders[5]. After Vickrey's (1961) seminal paper, auction theory was mainly developed in the 1980s. Although several issues were touched upon, such as the effects of risk aversion, correlation of information, budget constraints, asymmetries, and so forth, these were not felt as being the main issues in auction design in practice. In the 1990s, new models were developed that focused upon practical issues. Today, many economists regard auction theory as the best application of game theory to economics.

Auction theory is an important theory to study for several reasons. First, as many objects are being sold in auctions, it is important to understand how auctions work, and which auctions perform best, for instance in terms of generating revenues or in terms of efficiency. Second, auction theory is a fundamental tool in economic theory. It provides a price formation model, whereas the widely used Arrow-Debreu model from general equilibrium theory (Arrow and Debreu, 1954) is not explicit in how prices form. Also, the insights generated by auction theory can be useful when studying several other phenomena which have structures that resemble auctions, like lobbying contests, queues, war of attritions, and monopolist's market behavior (Klemperer, 2003). For instance, the theory of monopoly pricing is mathematically the same as the theory of revenue maximizing auctions (Bulow and Roberts, 1989). Reflecting its importance, auction theory has become a substantial field in economic theory.

This short paper gives an accessible introduction to the theory of single-object auctions. We focus on three types of questions. First, how much do bidders bid in equilibrium? Second, how much revenue do auctions raise? And third, which auction yields the highest expected revenue? The set-up of the paper is as follows. In Section *The Model*, we present the model (which is in fact Vickrey's symmetric independent private values model). In Section *Equilibrium bidding*, we derive the equilibrium bidding behavior for both the Dutch and the English auction (two auctions that are commonly used in practice) and calculate the expected revenue. In Section *Optimal auctions*, we discuss optimal auctions, i.e., auctions that maximize the expected revenue for the seller. The *revelation principle* and the *revenue equivalence theorem* do show that both the Dutch and the English auction are optimal if the seller imposes a reserve price. The last section concludes.

The model

We consider a situation with two risk-neutral non-colluding bidders who bid for one indivisible object. We assume that each bidder i has a value v_i for the object and that all these values are independently drawn from the uniform distribution on the interval [0,1]. The value v_i is private information to bidder i, and not known to the other bidders and the seller. The seller does not attach any value to the object. Furthermore, we assume that the bidders have unlimited budgets and that, if a bidder does not win the object, she is indifferent about who wins, and how much the winner pays. The auction being used (in Section *Equilibrium bidding*) is either the Dutch auction or the English auction. In Section *Optimal auctions*, the seller aims at finding a feasible auction mechanism which gives him the highest expected revenue.

Equilibrium bidding

In this section, we derive equilibrium bidding in the Dutch and the English auction, two auctions that are commonly used in practice. In the Dutch auction, the auctioneer begins with a very high price, and successively lowers it, until one bidder bids, i.e., announces that she is willing to accept the current price. This bidder wins the object at that price, unless the price is below the reserve price, i.e. the minimum price set by the seller. Flowers are sold this way in the Netherlands. It is not too difficult to prove that in equilibrium, each bidder bids half of her value:

Proposition 1 *Let $B_i(v_i) = 1/2\, v_i$, $i = 1,2$. Then (B_1,B_2) constitutes a symmetric Bayesian-Nash equilibrium of the Dutch auction. The expected revenue is equal to 1/3.*

Proof: If *bidding half your value* is a Bayesian-Nash equilibrium, it should be in both bidders' interest to follow this strategy if the other bidder does so as well. Let us suppose that bidder 2 bids $1/2v_2$, and see whether bidder 1 is indeed willing to bid $1/2v_1$. Of course, a priori, bidder 1 could bid any amount $b \geq 0$. Observe that a bid $b > 1/2$ does not make much sense for bidder 1 as she is always better off by bidding 1/2. The reason is that, regardless of her value, bidder 2 bids less than 1/2 because $v_2 \in [0,1]$. Therefore, bidder 1 always wins bidding b, but she could win at a lower price if she bids 1/2 instead. Her utility when she bids $b \in [0,1]$ is

$$P\left\{b \geq \frac{1}{2}v_2\right\}(v_1 - b) = P\{v_2 \leq 2b\}(v_1 - b) = 2b(v_1 - b)$$

Indeed, bidder 1 maximizes her utility by bidding $1/2v_1$. A similar reasoning holds true for bidder 2, so that *bidding half your value* is a Bayesian-Nash equilibrium. The expected revenue is equal to the expected value of $1/2v_{max}$, where $v_{max} = \max\{v_1,v_2\}$. As the expected value of the first-order statistic of two uniformly distributed variables over [0,1] is 2/3, the expected revenue is equal to 1/3.

In the English auction, the price starts at the reserve price, and is raised successively until one bidder remains. This bidder wins the object at the final price. The price can be raised by the auctioneer, or by having bidders call the bids themselves. The English auction is the most famous and most commonly used auction type. Art and wine are sold using this type of auction. We study here a version of the English auction called the Japanese auction, in which the price is raised continuously, and bidders announce to quit the auction at a certain price (e.g., by pressing or releasing a button). Obviously, each bidder stays in the auction up to the moment that the price reaches her value (we do not need any maths to show that this is true). After the first bidder quits the auction, the auction ends. The seller therefore receives in the English auction the second highest value. As the expected value of the lowest of two uniformly distributed variables over [0,1] is 1/3 the expected revenue is equal to 1/3.

Proposition 2 *Let $b_i(v_i) = v_i$, $i = 1,2$. Then b constitutes a Bayesian-Nash equilibrium of the English auction. In equilibrium, the expected revenue is equal to 1/3.*

Note that in both the Dutch and the English auction, the object always ends up in the hands of the bidder with the highest value as this is the bidder who submits the highest bid. Moreover, both auctions yield the same expected revenue for the seller. Natural questions that arise are: *Are there auctions that generate more revenue?* and *Which auction yields the highest expected revenue?* These questions will be answered in the next section.

Optimal auctions

In his remarkable paper, published in 1981, Myerson answers both questions in a model that includes our model as a special case[6]. In order to find the answers, Myerson derives two fundamental results, the *revelation principle* and the *revenue-equivalence theorem*. He starts by considering a special class of auctions: direct revelation games. In a direct revelation game, each bidder is asked to announce her value, and depending on the announcements, the object may or may be not allocated to one of the bidders, and each bidder may or may not have to pay a certain amount of money to the seller. A direct revelation game is incentive compatible (IC) if each bidder has an incentive to announce her value truthfully, and the game is individually rational (IR) if each bidder expects nonnegative utility.

Lemma 1 (Revelation Principle) *For any auction, there is an IC and IR direct revelation game that gives the seller the same expected equilibrium revenue as the auction.*

Proof: Consider an auction and the following direct revelation game. First, the seller asks each bidder to announce her value. Then, he determines the bid that each bidder would have chosen in the equilibrium of the auction given her announced value. Next, he implements the outcomes that would result in the auction from these bids. As the strategies form an equilibrium of the auction, it is an equilibrium for each bidder to announce her value truthfully in the direct revelation game. Therefore, the revelation game has the same outcome as the auction, so that both the seller and the bidders obtain the same expected utility as in the equilibrium of the auction.

Lemma 1 implies that when solving the seller's problem, there is no loss of generality in only considering direct revelation games that are individually rational and incentive compatible. Now, assume that in the optimal auction, if the seller sells the object, he always sells it to the bidder with the higher value[7]. Let $p(v_1)$ denote how much bidder 1 expects to pay in the equilibrium of an IC and IR direct revelation game if she announces that she has value v_1. Suppose, in contrast, that she announces that she has value w instead of v_1 and that bidder 2 does announce her value v_2 truthfully. Then bidder 1 only wins if her announced value is higher than bidder 2's true value, so that her expected utility is

$$U(v_1, w) = v_1 P\{v_2 \le w\} - p(w) = v_1 w - p(w) \tag{1}$$

In words: bidder 1's utility is (her value for the object) times (the probability that she wins) minus (her expected payments). Note that her winning probability and her expected payment depend on the value w that she announces, in contrast to her value for the object.

In equilibrium, bidder 1 should report her true value, so that (1) should be maximized at $w = v_1$:

$$\left.\frac{\partial U(v_1, w)}{\partial w}\right|_{w=v_1} = v_1 - p'(v_1) = 0$$

Integrating the above expression yields

$$p(v_1) = 1/2\ (v_1)^2 + c,$$

where c is a constant. In expectation, bidder 1 pays the following to the seller:

$$\int_{v^*}^{1} p(v_1)dv_1 = \int_{v^*}^{1}\left[\frac{1}{2}(v_1)^2 + c\right]dv_1 \tag{2}$$

where v^* is the value at which bidder 1 is indifferent between participating and not participating in the auction. Her expected utility in the auction is

$$U(v_1, v_1) = (v_1)^2 - p(v_1) = 1/2\ (v_1)^2 - c$$

so that she participates if and only if $c \le 1/2(v_1)^2$. This implies that

$$c = 1/2(v^*)^2 \tag{3}$$

Total expected revenue is the sum of bidder 1's and bidder 2's payment. Substituting (3) in (2), we find that bidder 1 expects to pay

$$EP_1(v^*) = \int_{v^*}^{1}\left[\frac{1}{2}(v_1)^2 + \frac{1}{2}(v^*)^2\right]dv_1$$

Because bidder 2 is identical to bidder 1, her expected payment $EP_2(v^*)$ is the same:

$$EP_2(v^*) = EP_1(v^*)$$

The seller's expected revenue $ER(v^*)$ can then be expressed as

$$ER(v^*) = EP_1(v^*) + EP_2(v^*)$$

$$= 2\int_{v^*}^{1}\left[\frac{1}{2}(v_1)^2 + \frac{1}{2}(v^*)^2\right]dv_1$$

$$= \int_{v}^{1}\left[(v_1)^2 + (v^*)^2\right]dv_1 \tag{4}$$

The following remarkable result arises from the above expression:

Proposition 3 (Revenue Equivalence Theorem) *Any auction that allocates the object to the bidder with the highest value, provided that this value exceeds v^*, yields the same expected revenue, which is given in (4).*

Myerson shows that the above version of the revenue equivalence theorem holds true for any number of bidders and for any *smooth* value distribution function. Indeed, our observation that the expected revenue in the Dutch auction and the English auction are the same follows immediately from this theorem: in equilibrium, both auctions allocate the object to the bidder with the highest value (so that $v^* = 0$).

Now, the seller maximizes his expected revenue by choosing the optimal v^* solving

$$\max_{v^*} ER(v^*) = \max_{v^*} \int_{v^*}^{1}\left[(v_1)^2 + (v^*)^2\right]dv_1$$

$$= \max_{v^*}\left[\frac{1}{3} + (v^*)^2 - \frac{4}{3}(v^*)^3\right]$$

The solution is $v^* = 1/2$. In other words, in the optimal auction, each bidder should only participate if her value is at least 1/2.

Proposition 4 *The optimal auction assigns the object to the bidder with the highest value, provided that the highest value exceeds 1/2. Otherwise, the seller keeps the object.*

It is not very difficult to show that both the Dutch auction and the English auction can implement the optimal auction if the seller imposes a reserve price equal to 1/2. The seller's expected revenue is 5/12 which is 25% higher than his revenue without a reserve price (which is 1/3 as we saw in the previous section). Myerson shows that Proposition 4 generalizes to any number of bidders, so that the optimal reserve price in both the Dutch and the English auction is 1/2, regardless of the number of bidders. Finally, Myerson proves that, under a mild restriction on the value distribution function, both the Dutch and the English auction implement the revenue maximizing auction with the correct reserve price.

Conclusion

In this paper we have derived the answer to the question how bidders bid in the Dutch and English auction under the assumption of independent private values. In the Dutch auction (in which the price moves downwards) it is optimal to stop the clock when the price reaches half your value (when the values are drawn from the uniform distribution); in the English auction (in which the price moves upwards) it is optimal to stay in the auction till the price reaches your value. We have seen that both auctions yield the same expected revenue. The revenue equivalence theorem tells us that this is no coincidence: any auction that allocates the object to the bidder with the highest value (provided that this value exceeds a certain threshold value) yields the same expected revenue. We have also derived the optimal (i.e. revenue-maximizing) auction. If the values are drawn from the uniform distribution ranging from 0 to 1, then the seller maximizes his revenue when he assigns the object to the bidder with the highest value provided that the highest value exceeds 1/2. Otherwise he keeps the object.

We should keep in mind that the revenue equivalence theorem is valid only under strict assumptions. Relaxing the assumptions of the symmetric independent private values model (as specified in Section 2) will lead to the collapse of the revenue equivalence theorem. We refer the reader to Maasland and Onderstal (2005) for a discussion of what happens to the revenue ranking of the Dutch and English auction when the assumptions are relaxed.

For the interested reader there are a couple of good survey articles and books available on auction theory. Early survey articles are McAfee and McMillan (1987), Matthews (1995), Wolfstetter (1996) and Klemperer (1999). Matthews (1995) is in particular relevant for those who are interested in the technicalities of the independent private values model. A more recent survey article is Maasland and Onderstal (2005). This article also contains an up-to-date survey of multi-object auctions. An overview of field studies on auctions can be found in Laffont (1997). Kagel (1995) presents a survey of laboratory experiments on auctions, while the books of Klemperer (2004) and Milgrom (2004) and the book chapter by Börgers and Van Damme (2004) discuss the use of auction theory in the design of real-life auctions. For an advanced treatment of auction theory we finally refer the reader to Krishna (2002).

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Notes

[1] See Cassady (1967) and Klemperer (2004).

[2] See McMillan (1994) and Cramton (1998).

[3] These so-called UMTS-auctions are studied in Jehiel and Moldavanu (2001), Klemperer (2002) and Van Damme (2002). Extensive analyses of the UMTS auctions in the UK and Germany can be found in Binmore and Klemperer (2002) and Grimm et al. (2002) respectively.

[4] William Vickrey earned the Nobel Memorial Prize in Economics in 1996 primarily for his work on auction theory.

[5] For instance, if a Van Gogh parinting is being auctioned and you want to buy it simply because you like it, then knowing how much your rivals value it would not affect how much you value it yourself.

[6] Independently, Riley and Samuelson (1981) derived similar results.

[7] Myerson (1981) shows that this is indeed the case in our setting.

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